

A GENERALIZATION OF THE CLUNIE–SHEIL-SMALL THEOREM

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ABSTRACT. In 1984, a simple and useful univalence criterion for harmonic functions was given by Clunie and Sheil-Small, which is usually called the shear construction. However, the application of this theorem is limited to the planar harmonic mappings convex in the horizontal direction. In this paper, a natural generalization of the shear construction is given. More precisely, our results are obtained under the hypothesis that the image of a harmonic mapping is a sum of two sets convex in the horizontal direction.

1. INTRODUCTION

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane \mathbb{C} . A function $f : \mathbb{D} \rightarrow \mathbb{C}$ is said to be harmonic, if its real and imaginary parts are real harmonic, i.e. they satisfy the Laplace equation. Since \mathbb{D} is simply connected it is well-known that f can be written in the form

$$(1.1) \quad f(z) = h(z) + \overline{g(z)}, \quad z \in \mathbb{D},$$

where h and g are analytic in \mathbb{D} . The Jacobian J_f of f in terms of h and g is given by

$$(1.2) \quad J_f(z) = |h'(z)|^2 - |g'(z)|^2, \quad z \in \mathbb{D}.$$

Among all the harmonic functions in \mathbb{D} one can distinguish those with non-vanishing Jacobian. In fact, it is proved that such harmonic functions are locally 1-1. If the Jacobian of a harmonic function in \mathbb{D} is positive, it means that this function is locally 1-1 and sense-preserving. More information about basics of harmonic functions can be found e.g. in [2].

Clunie and Sheil-Small in [1] gave the following theorem, known as the shear construction.

Theorem A. *A function $f = h + \bar{g}$ harmonic in \mathbb{D} with positive Jacobian is 1-1 sense-preserving mapping of \mathbb{D} onto a domain convex in the direction of the real axis if, and only if, $h - g$ is an analytic 1-1 mapping of \mathbb{D} onto a domain convex in the direction of the real axis.*

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It appeared to have many applications as an univalence criterion and as a method of constructing harmonic mappings (see, e.g., [4, 5, 6, 7, 8, 9, 10, 12]).

In this paper we generalize the theorem of Clunie and Sheil-Small. In Section 2 we show some auxiliary results. In Section 3 we use results from Section 2 to give new conditions for univalence of the planar harmonic mappings.

2. TOPOLOGICAL PROPERTIES

The proof of Theorem A of Clunie and Sheil-Small relies on the following lemma, which will be also useful in our considerations.

Lemma B. *Let D be a domain convex in the direction of the real axis and let p be a continuous real-valued function in D . Then the mapping $D \ni w \mapsto w + p(w)$ is 1-1 in D if, and only if, it is locally 1-1. In this case the image of D is convex in the direction of the real axis.*

Using this lemma we will prove more general results and apply them to obtain new univalence criteria for harmonic mappings. For a given set D in the complex plane \mathbb{C} it will be convenient to define

$$(2.1) \quad P_y(D) := \{a \in \mathbb{R} : \exists_{z \in D} \operatorname{Im} z = a\}.$$

Such defined set $P_y(D)$ has several immediate properties, which we formulate in the following lemma for convenience in further use.

Lemma 2.1. *Let D_1 and D_2 be the domains with nonempty intersection such that $D_1 \cup D_2$ is simply connected. Then $P_y(D_1)$, $P_y(D_2)$ and $P_y(D_1 \cap D_2)$ are open intervals.*

Proof. The Janiszewski theorem [11, p. 268, Theorem 2] yields the connectedness of the set $D_1 \cap D_2$, which clearly is also open. Thus $D_1 \cap D_2$ is a nonempty domain as well as D_1 and D_2 . Hence, obviously, $P_y(D_1)$, $P_y(D_2)$ and $P_y(D_1 \cap D_2)$ are open and connected subset of the real line \mathbb{R} , which completes the proof. \square

Using this lemma we can prove the following theorem.

Theorem 2.2. *Let D_1 , D_2 be the domains convex in the direction of the real axis and let $q : D_1 \cup D_2 \rightarrow \mathbb{C}$ be a continuous function for which Jacobian J_q exists, and such that $\operatorname{Im} q(z) = \operatorname{Im} z$ for all $z \in D_1 \cup D_2$. Then q is 1-1 if, and only if, $J_q \neq 0$ and*

$$P_y(D_1 \cap D_2) = P_y(q(D_1) \cap q(D_2)).$$

Proof. If D_1, D_2 are two disjoint domains convex in the direction of the real axis then our claim follows immediately from Theorem A. Hence, we consider the case $D_1 \cap D_2 \neq \emptyset$.

Assume that q is 1-1 in $D_1 \cup D_2$. We show that Jacobian is not equal to 0 and $P_y(D_1 \cap D_2) = P_y(q(D_1) \cap q(D_2))$. It is clear that if q is 1-1, then it is locally 1-1 and thus $J_q \neq 0$. It is also clear that $P_y(D_1 \cap D_2) \subset P_y(q(D_1) \cap q(D_2))$. We show the inverse inclusion. Let $a \in \mathbb{R} \setminus P_y(D_1 \cap D_2)$ be fixed. Then for any choice of $z_1 \in D_1$ and $z_2 \in D_2$ such that $\text{Im } z_1 = a$ and $\text{Im } z_2 = a$ we have $q(z_1) \neq q(z_2)$, since q is 1-1. Thus we deduce that $a \notin P_y(q(D_1) \cap q(D_2))$, which means that $P_y(q(D_1) \cap q(D_2)) \subset P_y(D_1 \cap D_2)$. Hence we get $P_y(D_1 \cap D_2) = P_y(q(D_1) \cap q(D_2))$.

To prove the converse we assume that $J_q \neq 0$ and $P_y(D_1 \cap D_2) = P_y(q(D_1) \cap q(D_2))$ and we show that q is 1-1. The property $\text{Im } q(z) = \text{Im } z$ for all $z \in D_1 \cup D_2$ together with Lemma B ensure that q is 1-1 in $(D_1 \cup D_2) \cap \{z \in \mathbb{C} : \text{Im } z \in P_y(D_1 \cap D_2)\}$. Assume that q be not 1-1 in

$$\tilde{D} := (D_1 \cup D_2) \cap \{z \in \mathbb{C} : \text{Im } z \notin P_y(D_1 \cap D_2)\}.$$

Then, there exist $a \in \tilde{D}$ and $z_1, z_2 \in D_1 \cup D_2$ such that $a = \text{Im } z_1 = \text{Im } z_2$ and $q(z_1) = q(z_2)$. But the last equality means that $a \in P_y(q(D_1) \cap q(D_2))$ and by the definition of \tilde{D} we have $a \notin P_y(D_1 \cap D_2)$, which is a contradiction to the assumption that $P_y(D_1 \cap D_2) = P_y(q(D_1) \cap q(D_2))$. Thus, q is 1-1 in \tilde{D} . Now, the property $\text{Im } q(z) = \text{Im } z$ for all $z \in D_1 \cup D_2$ implies that q is 1-1 in $D_1 \cup D_2$ and this completes the proof. \square

Replacing the univalence condition in Theorem 2.2 by the condition that the sets $D_1 \cup D_2$ and $q(D_1) \cup q(D_2)$ are simply connected we get.

Theorem 2.3. *Let D_1, D_2 be the domains convex in the direction of the real axis with nonempty intersection and let $q : D_1 \cup D_2 \rightarrow \mathbb{C}$ be a continuous function, such that J_q exists and it is not equal to 0, and $\text{Im } q(z) = \text{Im } z$ for all $z \in D_1 \cup D_2$. If $D_1 \cup D_2$ and $q(D_1) \cup q(D_2)$ are simply connected then*

$$P_y(D_1 \cap D_2) = P_y(q(D_1) \cap q(D_2)).$$

Proof. First, observe that the inclusion

$$(2.2) \quad P_y(D_1 \cap D_2) \subset P_y(q(D_1) \cap q(D_2))$$

is valid for all domains D_1, D_2 and for all functions q satisfying assumptions of Theorem 2.3.

Now, we prove the inverse inclusion. We can assume that the Jacobian J_q is positive. Notice, that if $D_1 \cup D_2$ is simply connected then, by Lemma 2.1, the set $P_y(D_1 \cap D_2)$ is connected. By the same Lemma 2.1 and obvious equality $q(D_1 \cup D_2) = q(D_1) \cup q(D_2)$

we deduce that $P_y(q(D_1) \cap q(D_2))$ is connected since $q(D_1 \cup D_2)$ is simply connected. Moreover, $P_y(D_1 \cap D_2)$ and $P_y(q(D_1) \cap q(D_2))$ are open since $D_1 \cap D_2$ and $q(D_1) \cap q(D_2)$ are open, hence $P_y(D_1 \cap D_2)$ and $P_y(q(D_1) \cap q(D_2))$ are nonempty open intervals.

Next, assume that there exists a real number a such that $a \in P_y(q(D_1) \cap q(D_2))$ and $a \notin P_y(D_1 \cap D_2)$. Then, there exist $\tilde{a} \in P_y(q(D_1) \cap q(D_2)) \setminus P_y(D_1 \cap D_2)$ and $\varepsilon > 0$ such that the sets

$$\tilde{A}_\varepsilon := (\tilde{a} - \varepsilon, \tilde{a} + \varepsilon) \cap P_y(D_1 \cap D_2) \quad \text{and} \quad (\tilde{a} - \varepsilon, \tilde{a} + \varepsilon) \setminus (P_y(D_1 \cap D_2) \cup \{\tilde{a}\})$$

are nonempty open intervals. Indeed, this follows from the properties of $P_y(D_1 \cap D_2)$ and $P_y(q(D_1) \cap q(D_2))$ as the open and nonempty intervals. Now, since $\tilde{a} \in P_y(q(D_1) \cap q(D_2))$ and $q(D_1) \cap q(D_2)$ is open, we can find points $w_1, w_2 \in q(D_1) \cap q(D_2)$ such that

$$\operatorname{Re} w_1 < \operatorname{Re} w_2 \quad \text{and} \quad \operatorname{Im} w_1 = \operatorname{Im} w_2 = \tilde{a}.$$

Recall, that $\tilde{a} \notin P_y(D_1 \cap D_2)$, thus there exist points

$$\eta_1, \eta_2 \in D_1 \quad \text{and} \quad \zeta_1, \zeta_2 \in D_2$$

such that $q(\eta_1) = q(\zeta_1) = w_1$ and $q(\eta_2) = q(\zeta_2) = w_2$ and, by Lemma B, they are unique. Moreover, the assumption that the Jacobian J_q is positive implies either

$$\operatorname{Re} \eta_1 < \operatorname{Re} \eta_2 < \operatorname{Re} \zeta_1 < \operatorname{Re} \zeta_2,$$

or

$$\operatorname{Re} \zeta_1 < \operatorname{Re} \zeta_2 < \operatorname{Re} \eta_1 < \operatorname{Re} \eta_2.$$

Now, since D_1 and D_2 are open sets and \tilde{A}_ε is a nonempty, open interval, then there exist sequences

$$\mathbb{N} \ni n \mapsto \eta_{1,n} \in D_1, \quad \eta_{1,n} \rightarrow \eta_1, \quad \text{and} \quad \mathbb{N} \ni n \mapsto \eta_{2,n} \in D_1, \quad \eta_{2,n} \rightarrow \eta_2,$$

and the sequences

$$\mathbb{N} \ni n \mapsto \zeta_{1,n} \in D_2, \quad \zeta_{1,n} \rightarrow \zeta_1 \quad \text{and} \quad \mathbb{N} \ni n \mapsto \zeta_{2,n} \in D_2, \quad \zeta_{2,n} \rightarrow \zeta_2,$$

with $\operatorname{Im} \eta_{1,n} = \operatorname{Im} \eta_{2,n} = \operatorname{Im} \zeta_{1,n} = \operatorname{Im} \zeta_{2,n} \in \tilde{A}_\varepsilon$ and such that either

$$\operatorname{Re} \eta_{1,n} < \operatorname{Re} \eta_{2,n} < \operatorname{Re} \zeta_{1,n} < \operatorname{Re} \zeta_{2,n},$$

(2.3) or

$$\operatorname{Re} \zeta_{1,n} < \operatorname{Re} \zeta_{2,n} < \operatorname{Re} \eta_{1,n} < \operatorname{Re} \eta_{2,n},$$

for sufficiently large n . Next, from the continuity of q we deduce that

$$q(\eta_{1,n}) \rightarrow w_1, \quad q(\eta_{2,n}) \rightarrow w_2 \quad \text{and} \quad q(\zeta_{1,n}) \rightarrow w_1, \quad q(\zeta_{2,n}) \rightarrow w_2.$$

Thus, by the assumption that $J_q > 0$ and (2.3) we have either

$$\operatorname{Re} q(\eta_{2,n}) < \operatorname{Re} q(\zeta_{1,n}) \quad \text{or} \quad \operatorname{Re} q(\zeta_{2,n}) < \operatorname{Re} q(\eta_{1,n}),$$

for sufficiently large n , which implies $\operatorname{Re} w_2 \leq \operatorname{Re} w_1$. But this is a contradiction to the assumption $\operatorname{Re} w_1 < \operatorname{Re} w_2$. Thus, we have the inclusion $P_y(q(D_1) \cap q(D_2)) \subset P_y(D_1 \cap D_2)$, which together with (2.2) yields $P_y(D_1 \cap D_2) = P_y(q(D_1) \cap q(D_2))$, and this completes the proof. \square

Corollary 2.4. *Let D_1, D_2 be the domains convex in the direction of the real axis with nonempty intersection, such that $D_1 \cup D_2$ is simply connected and let $q : D_1 \cup D_2 \rightarrow \mathbb{C}$ be a continuous function for which the Jacobian J_q exists, such that $\operatorname{Im} q(z) = \operatorname{Im} z$ for all $z \in D_1 \cup D_2$ and $q(D_1) \cup q(D_2)$ is simply connected. Then $J_q \neq 0$ if, and only if, q is 1-1.*

Proof. It is an immediate consequence of Theorem 2.2 and Theorem 2.3. \square

3. HARMONIC MAPPINGS

In this section we apply the results obtained in the previous section to the theory of harmonic mappings. We start with the definition which will simplify our considerations. For a given set D let

$$(3.1) \quad \Lambda_y(D) := \{a \in \mathbb{R} : (D \cap \{z \in \mathbb{C} : \operatorname{Im} z = a\}) \text{ is a nonempty and connected set}\}.$$

We will see the set Λ_y is as much convenient in the following investigations as the set P_y , defined by (2.1), was in the previous section. Thus, we need the following lemma describing a connection between P_y and Λ_y .

Lemma 3.1. *Let D_1, D_2 be the domains convex in the direction of the real axis with nonempty intersection. Then $P_y(D_1 \cap D_2) = \Lambda_y(D_1 \cup D_2)$.*

Proof. Let D_1, D_2 be the domains convex in the direction of the real axis with nonempty intersection. We will show both inclusions

$$P_y(D_1 \cap D_2) \subset \Lambda_y(D_1 \cup D_2) \quad \text{and} \quad \Lambda_y(D_1 \cup D_2) \subset P_y(D_1 \cap D_2).$$

Assume first, that $a \in P_y(D_1 \cap D_2)$. Then, there exists $w \in D_1 \cap D_2$ such that $\operatorname{Im} w = a$. This means, that

$$w \in (D_1 \cap \{z \in \mathbb{C} : \operatorname{Im} z = a\}) \cap (D_2 \cap \{z \in \mathbb{C} : \operatorname{Im} z = a\}).$$

Next, observe that the sets $D_1 \cap \{z \in \mathbb{C} : \operatorname{Im} z = a\}$ and $D_2 \cap \{z \in \mathbb{C} : \operatorname{Im} z = a\}$ are nonempty and connected, since both domains D_1 and D_2 are convex in the direction of

the real axis, and in addition they have nonempty intersection. Thus, the set

$$(D_1 \cup D_2) \cap \{z \in \mathbb{C} : \operatorname{Im} z = a\}$$

is nonempty and connected, and consequently $a \in \Lambda_y(D_1 \cup D_2)$.

Now, we prove the second inclusion. Let $a \in \Lambda_y(D_1 \cup D_2)$, then the set

$$(D_1 \cup D_2) \cap \{z \in \mathbb{C} : \operatorname{Im} z = a\}$$

is nonempty and connected. Next, observe that

$$D_1 \cap \{z \in \mathbb{C} : \operatorname{Im} z = a\} \quad \text{and} \quad D_2 \cap \{z \in \mathbb{C} : \operatorname{Im} z = a\}$$

are open and connected intervals since D_1 and D_2 are open and convex in the direction of the real axis. Hence, there exists $w \in D_1 \cap D_2$, such that $\operatorname{Im} w = a$ and thus, $a \in P_y(D_1 \cap D_2)$, which completes the prove. \square

Now, we can apply results obtained in Section 2 to harmonic mappings.

Theorem 3.2. *Let $f = h + \bar{g}$ be a harmonic function in \mathbb{D} such that $J_f > 0$ in \mathbb{D} . If $\Lambda_y((h - g)(\mathbb{D})) = \Lambda_y(f(\mathbb{D}))$ then the following statements are equivalent*

- (1) *f is 1-1 mapping and $f(\mathbb{D})$ is a sum of two non-disjoint domains convex in the direction of the real axis.*
- (2) *$h - g$ is 1-1 analytic mapping and $(h - g)(\mathbb{D})$ is a sum of two non-disjoint domains convex in the direction of the real axis.*

Proof. Let $f = h + \bar{g}$ be a harmonic function in the unit disk and such that J_f is positive in \mathbb{D} , and let $\Lambda_y((h - g)(\mathbb{D})) = \Lambda_y(f(\mathbb{D}))$. We show that (1) \Rightarrow (2) and (2) \Rightarrow (1).

(1) \Rightarrow (2). Assume that f is 1-1 in the unit disk and that $f(\mathbb{D}) = D_1 \cup D_2$, where $D_1, D_2 \subset \mathbb{C}$ are domains convex in the direction of the real axis with nonempty intersection. Then there exists $f^{-1} : D_1 \cup D_2 \rightarrow \mathbb{D}$ and the composition $q := (h - g) \circ f^{-1}$ is well defined continuous function. Observe, that $q(w) = (h - g)(f^{-1}(w)) = w - 2 \operatorname{Re} g(f^{-1}(w))$ for all $w \in D_1 \cup D_2$. Moreover, by Lemma 3.1 we have

$$(3.2) \quad \Lambda_y(f(\mathbb{D})) = \Lambda_y(D_1 \cup D_2) = P_y(D_1 \cap D_2).$$

Similarly, by Lemma 3.1 and by equality $q(D_1 \cup D_2) = q(D_1) \cup q(D_2)$ we have

$$(3.3) \quad \Lambda_y((h - g)(\mathbb{D})) = \Lambda_y(q(D_1 \cup D_2)) = \Lambda_y(q(D_1) \cup q(D_2)) = P_y(q(D_1) \cap q(D_2)).$$

The formulae (3.2) and (3.3), together with the hypothesis $\Lambda_y((h - g)(\mathbb{D})) = \Lambda_y(f(\mathbb{D}))$, yield

$$(3.4) \quad P_y(D_1 \cap D_2) = P_y(q(D_1) \cap q(D_2)).$$

Thus, the assumptions of Theorem 2.2 are satisfied and in consequence we obtain that q is 1-1 in \mathbb{D} . Hence, $h - g$ is 1-1 in \mathbb{D} , since f is. Additionally, both sets $q(D_1)$ and $q(D_2)$ are domains convex in the direction of the real axis, by Lemma B, and their intersection is not empty by (3.4).

(2) \Rightarrow (1). Now, assume that $h - g$ is 1-1 in the unit disk and that $(h - g)(\mathbb{D}) = \Omega_1 \cup \Omega_2$, where $\Omega_1, \Omega_2 \subset \mathbb{C}$ are domains convex in the direction of the real axis with nonempty intersection. Then there exists $(h - g)^{-1} : \Omega_1 \cup \Omega_2 \rightarrow \mathbb{D}$ and the composition $q := f \circ (h - g)^{-1}$ is well defined continuous function. Observe, that we have $q(w) = f((h - g)^{-1}(w)) = w + 2 \operatorname{Re} g((h - g)^{-1}(w))$ for all $w \in \Omega_1 \cup \Omega_2$. Reasoning similar to the one used in previous case and the use of Lemma 3.1 give us equality

$$(3.5) \quad P_y(\Omega_1 \cap \Omega_2) = P_y(q(\Omega_1) \cap q(\Omega_2)).$$

Again, the assumptions of Theorem 2.2 are satisfied and in consequence we obtain that q is 1-1 in \mathbb{D} , thus f is 1-1 in \mathbb{D} , since $h - g$ is. Finally, $q(\Omega_1)$ and $q(\Omega_2)$ are domains convex in the direction of the real axis, by Lemma B, and their intersection is not empty by (3.5). \square

As a consequence of Theorem 3.2 we obtain a generalization of Theorem A of Clunie and Sheil-Small.

Theorem 3.3. *Let $f = h + \bar{g}$ be a harmonic function in \mathbb{D} such that $J_f > 0$ in \mathbb{D} . If $(h - g)(\mathbb{D})$ and $f(\mathbb{D})$ are nonempty simply connected domains then the following statements are equivalent*

- (1) *f is 1-1 mapping and $f(\mathbb{D})$ is a sum of two non-disjoint domains convex in the direction of the real axis.*
- (2) *$h - g$ is 1-1 analytic mapping and $(h - g)(\mathbb{D})$ is a sum of two non-disjoint domains convex in the direction of the real axis.*

Proof. Observe, that if f is 1-1 in \mathbb{D} and $f(\mathbb{D}) = D_1 \cup D_2$, where $D_1, D_2 \subset \mathbb{C}$ are domains convex in the direction of the real axis with nonempty intersection, then the function

$$D_1 \cup D_2 \ni w \mapsto q_f(w) := (h - g)(f^{-1}(w)) = w - 2 \operatorname{Re} g(f^{-1}(w))$$

is well-defined and continuous in $D_1 \cup D_2$. The same is true if we assume that $h - g$ is 1-1 in \mathbb{D} and $(h - g)(\mathbb{D}) = \Omega_1 \cup \Omega_2$, where $\Omega_1, \Omega_2 \subset \mathbb{C}$ are domains convex in the direction of the real axis with nonempty intersection, that is the function

$$D_1 \cup D_2 \ni w \mapsto q_{h-g}(w) := f((h - g)^{-1}(w)) = w + 2 \operatorname{Re} g((h - g)^{-1}(w))$$

is well-defined and continuous in $\Omega_1 \cup \Omega_2$.

Since $(h - g)(\mathbb{D})$ and $f(\mathbb{D})$ are nonempty simply connected domains then by Theorem 2.3 and Lemma 3.1, the proof follows from Theorem 3.2. \square

If one omits in Theorem 3.3 the assumption that both $f(\mathbb{D})$ and $(h - g)(\mathbb{D})$ are simply connected, then the Theorem 3.3 is no longer true which is shown in the following example.

Example 3.4. Consider vertical shear of the rotated Koebe function with dilatation $\omega(z) := iz$. From the equations

$$\begin{aligned} h(z) - g(z) &= \frac{z}{(1 - iz)^2} \\ g'(z) &= izh'(z) \end{aligned}$$

we get

$$\begin{aligned} h(z) &= \frac{-6iz - 3z^2 + iz^3}{6(i + z)^3}, \\ g(z) &= \frac{3z^2 + iz^3}{6(i + z)^3}, \end{aligned}$$

and

$$f(z) = h(z) + \overline{g(z)} = \frac{-6iz - 3z^2 + iz^3}{6(i + z)^3} + \overline{\left(\frac{3z^2 + iz^3}{6(i + z)^3} \right)}.$$

Now, using transformation

$$w = u + iv := \frac{1 + iz}{1 - iz},$$

which maps the unit disk onto the right half-plane, i.e. $\{w \in \mathbb{C} : \operatorname{Re} w > 0\}$ we get

$$\begin{aligned} h(z) - g(z) &= \frac{1}{4i}(w^2 - 1), \\ h(z) + g(z) &= \frac{1}{6i}(w^3 - 1), \end{aligned}$$

and consequently

$$f(z) = \operatorname{Re}(h(z) + g(z)) + i \operatorname{Im}(h(z) - g(z)) = -\frac{1}{6} \operatorname{Im}(w^3 - 1) - \frac{i}{4} \operatorname{Re}(w^2 - 1).$$

After some calculations we obtain

$$(3.6) \quad f(z) = -\frac{1}{6}v(3u^2 - v^2) - \frac{i}{4}(u^2 - v^2 - 1),$$

where $u > 0$ and $v \in \mathbb{R}$.

Clearly, the function $h(z) - g(z)$ maps the unit disk onto the plane with the slit along the imaginary axis, more precisely onto $\mathbb{C} \setminus \{z \in \mathbb{C} : \operatorname{Im} z \geq \frac{1}{4} \text{ and } \operatorname{Re} z = 0\}$, which is a simply connected domain. On the other hand, the formula (3.6), allows us to find the image of the unit disk via the map $f(z)$, by studying which parts of the vertical lines of the complex plane belong to $f(\mathbb{D})$. First, observe that $\operatorname{Re} f(z) = 0$ if and only if $v = 0$ or $v^2 = 3u^2$. Thus, we have $\operatorname{Im} f(z) = \frac{1}{4} - \frac{u^2}{4}$, with $u > 0$, if $v = 0$ and $\operatorname{Im} f(z) = \frac{1}{4} + \frac{u^2}{2}$, with $u > 0$, if $v^2 = 3u^2$, and consequently we get that the point $\frac{i}{4}$ do not belong to $f(\mathbb{D})$.

Now, assume that $\operatorname{Re} f(z) = c$ with $c \neq 0$. Then, since $v \neq 0$, we have $u^2 = \frac{v^2}{3} - \frac{2c}{v}$ and

$$\operatorname{Im} f(z) = \frac{2v^3 + 3v + 6c}{12v}, \quad \text{where } v \in (-\infty, 0) \cup (0, +\infty).$$

If $c > 0$ and $v \in (-\infty, 0)$ then

$$\begin{aligned} \lim_{v \rightarrow -\infty} \frac{2v^3 + 3v + 6c}{12v} &= +\infty, \\ \lim_{v \rightarrow 0^-} \frac{2v^3 + 3v + 6c}{12v} &= -\infty, \end{aligned}$$

and the whole vertical line $w = c$ belongs to $f(\mathbb{D})$. Analogously, if $c < 0$ and $v \in (0, +\infty)$ then

$$\begin{aligned} \lim_{v \rightarrow +\infty} \frac{2v^3 + 3v + 6c}{12v} &= +\infty, \\ \lim_{v \rightarrow 0^+} \frac{2v^3 + 3v + 6c}{12v} &= -\infty, \end{aligned}$$

and in that case the whole vertical line $w = c$ belongs to $f(\mathbb{D})$, too. Hence, we get $f(\mathbb{D}) = \mathbb{C} \setminus \{\frac{i}{4}\}$ which is not simply connected domain. The function f fail to satisfy assumptions of Theorem 3.2, and straightforward calculations shows that $f(\frac{\sqrt{3}}{3}) = f(-\frac{\sqrt{3}}{3}) = \frac{3i}{8}$, thus f is not univalent in \mathbb{D} .

Remark 3.5. *Recall, that Theorem A can be reformulated and it remains valid for a function convex in any fixed direction. Notice, that our results can also be rewritten in this fashion.*

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